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Production Function

A production function is the relationship by which inputs are combined to produce an output. It indicates the highest output that a firm can produce for every specified combination of inputs. If we focus on two inputs labour and capital then production can be written as

$$Y = f(L, K)$$

This equation applies to a given technology – i.e. given state of knowladge about various methods that might be used to transform inputs into outputs. In the figure we use only input 'L' and one output measured by q. All the dotted region and boundary points show technically feasible level of outputs. But the boundary of this production set is known as production function, as it shows maximum output that can be obtained on a given input level.

Short Run and Long Run Production Function

Short Run	Long Run
i) Most of the inputs are fixed. Only one or a few of the inputs can be changed.	i) All factors are variable
ii) There is always a concept of fixed cost.	ii) No fixed cost.
iii) S.R. pdn func. can be written as $Q = f(L, \bar{K})$	iii) L.R. pdn. func. $Q = f(L, K)$
iv) There is returns to a variable factor.	iv) There is Returns to scale.

**[The quantity of one factor ~~(productive service)~~ is increased by equal amounts, the quantities of other factors remaining fixed, the ~~resulting~~ increment of TP will first increase but decrease after some point.]

Law of Diminishing Returns to Factors or Law of Variable proportions

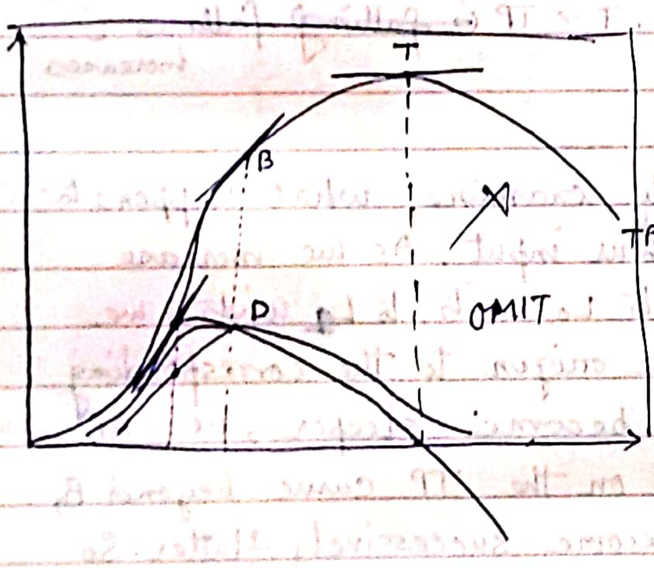
According to the law of diminishing marginal returns to factors, when one or more inputs are fixed, a variable input is likely to have a marginal product that eventually diminishes as the level of input increases after some point. It states that with a given state of technology if **

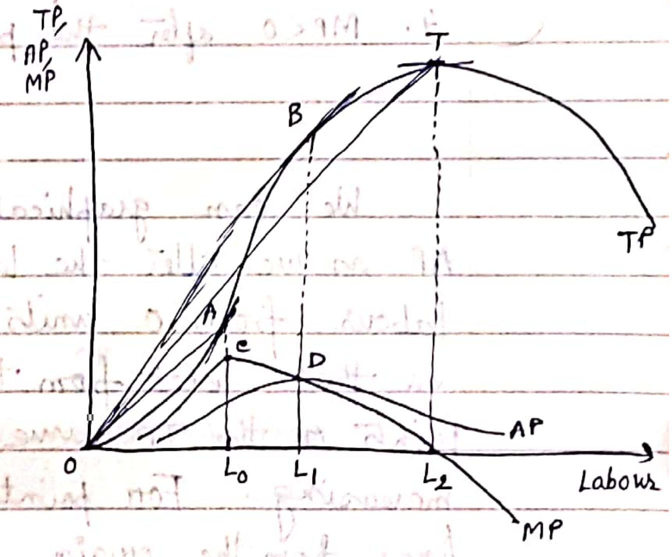
Production with a Single Variable Input: Total, Average and Marginal Products

Total output is usually called total product by economists and is denoted by TP. We can derives the marginal product MP, which is the increase in total product due to a one-unit increase in Labour or $\frac{\Delta TP}{\Delta L}$ or $\frac{\Delta Q}{\Delta L}$. We can ~~do~~ calculate average product which is simply output per unit of labour or $\frac{TP}{L}$ or $\frac{Q}{L}$.

Land	Lab.	TP	MP	AP
10	1	5	5	5
10	2	14	9	7
10	3	30	16	10
10	4	52	22	13
10	5	70	18	14
10	6	84	14	14
10	7	91	7	13
10	8	96	5	12
10	9	96	0	10.6
10	10	90	-6	9

In this example, the amount of land is fixed at 10 acres. The employment of lab. is increasing. When, at first, the number of workers increases from 1 to 2, production increases from 5 tonnes to 14 tonnes i.e. MP_L is 9 tonnes; when a third worker is employed, ~~(is employed)~~ output is 30 tonnes. MP_L is now 16 tonnes. Since MP has increased, the law of increasing returns to lab. is in force here. After the employment of 4-th unit of lab., ~~mo~~ MP_L falls. When the no. of workers is nine, TP is 96 tonnes and it reaches its maximum. The MP_L is zero in that case. Similarly when there are 10 workers, the MP is negative, because total product begins to fall from that stage.





In the figure the amount of the factor has again been measured on the horizontal axis. Total, marginal and average products have been measured on the vertical axis. The curve that shows total output at various levels of employment of the factor is called the Total Product (TP) curve. At first when MP is increasing, TP increases at an increasing rate. Later there comes a stage when total product increases increases at a diminishing rate i.e. marginal product decreases. Up to the pt. C, the MP increases, hence TP increases at an

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increasing rate (at L_0 level). After that pt. it diminishes. At L_2 level of factor employment, TP curve reaches its maximum (at pt. T). Thus corresponding to that level of employment, MP=0. So the MP cuts the base at L_2 level of employment.

MP is equal to $\Delta TP/\Delta L$. So graphically it is the slope of the ~~tangent~~ TP curve.

Relation betn TP & MP

1. MP > 0 and increasing (0 to c) → TP increases at an increasing rate as L increases
2. MP>0 and decreasing (c to T) → TP increases at an decreasing rate as L increase
3. MP=0 at the pt. T → TP is constant as L increases
4. MP<0 after the pt. T → TP ~~is falling~~ falls as L increases

We can graphically examine what happens to AP as we alter the labour input. As we increase labour from 0 units to L_0 units to L_1 units, we see that lines from the origin to the corresponding points on the TP curve become steeper, i.e., AP increasing. For points on the TP curve beyond B, lines from the origin become successively flatter. So beyond L_1 units of lab., AP is decreasing. Now if AP is increasing to the left of B and decreasing to the right of B, AP must reach its maximum at L_1 units of labour corresponding to the pt. B on the TP curve. So AP is maximized at a quantity corresponding to the point of tangency between the TP curve and a line from the origin. It can be shown that, it is at this highest pt. D of the AP curve that the MP curve will intersect

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the AP curve.

MP → the slope of the TP curve

AP – slope of the line joining the origing to the corresponding point ~~of~~ on the TP curve.

Slope of the AP curve $\left(\frac{dAP_L}{dL}\right)$ may be positive, zero or negative,

$$\frac{d(AP_L)}{dL} \gtreqless 0.$$

Now, $AP_L = \frac{q}{L}$

$$\frac{d(AP_L)}{dL} = \frac{\frac{dq}{dL}\cdot L - q}{L^2} = \frac{1}{L}\left[\frac{dq}{dL} - \frac{q}{L}\right]$$

$$= \frac{1}{L}\left[MP_L - AP_L\right]$$

Case I – As $MP_L > AP_L$

$\Rightarrow \frac{d(AP_L)}{dL} > 0$ i.e. AP_L is rising

Again, $MP_L > AP_L \rightarrow \frac{\Delta q}{\Delta L} > \frac{q}{L}$ or, $\frac{\Delta q}{q} > \frac{\Delta L}{L}$

i.e. Increasing Returns to factor

Case II – As $MP_L < AP_L$

$\Rightarrow \frac{d(AP_L)}{dL} < 0$ i.e. AP_L is falling

Again $MP_L < AP_L \Rightarrow \frac{\Delta q}{\Delta L} < \frac{q}{L} \Rightarrow \frac{\Delta q}{q} < \frac{\Delta L}{L}$

i.e. Decreasing Returns to factor

Case III – As $MP_L = AP_L$

$\Rightarrow \frac{d(AP_L)}{dL} = 0 \Rightarrow AP_L$ is maximum.

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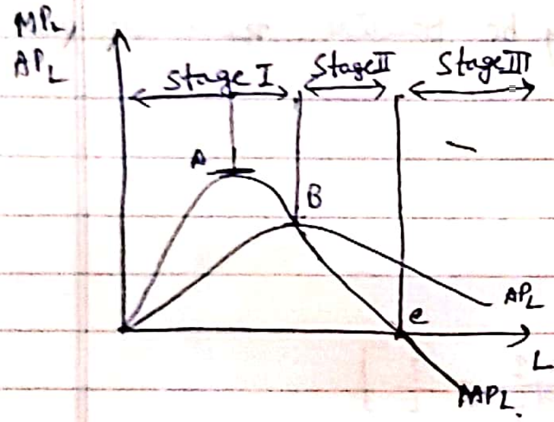
If MP is more than the AP, then the contribution to output of an additional unit of variable input exceeds the average contribution to the variable used thus far, the average contribution must rise.



Stage I → $MP_L > AP_L$ ⇒ Increasing Returns to factor
Stage II → $MP_L < AP_L$ ⇒ Decreasing Returns to factor
Stage III → $MP_L < 0$ ⇒ TP is falling ⇒ Negative Returns.

No profit maximizing producer would produce in Stage I or III. In stage I by adding one more unit of Labour, the producer can increase the average productivity of all the units. Thus it would be unwise on the part of the producer to stop pdn at this stage. As for stage III, it does not pay the producer to be in this region because by reducing the lab. input she can increase total output and save the cost of a unit of Labour. Thus, economically meaningful range is just that given by stage II.

The profit of the firm by using a single variable input is given by

$$\pi = P.q - W.L - \bar{F}$$

$$\frac{d\pi}{dL} = P.\frac{dq}{dL} - W = 0$$

$$\Rightarrow P.MP_L = W$$

$$\Rightarrow MP_L = \frac{W}{P} \quad \ldots\ldots (1)$$

$$\frac{d^2\pi}{dL^2} = P.\frac{d^2q}{dL^2} < 0.$$

$$\Rightarrow \frac{d^2q}{dL^2} < 0$$

$\Rightarrow$ MP_L is falling

In S.R. firm will continue to produce if $\pi(q>0)$ is greater than $\pi(q=0)$

$\therefore$ $Pq - WL - \bar{F} > -\bar{F}$ $\qquad$ $\pi = Pq - WL - \bar{F}$ (when $q>0$)

$\Rightarrow$ $Pq - WL > 0$

$\Rightarrow$ $Pq > WL$ $\qquad$ $\pi = -\bar{F}$ (when $q=0$, firm incurs loss which is equal to fixed cost i.e. $\bar{F}$)

$\Rightarrow$ $\frac{q}{L} > \frac{W}{P}$

$\Rightarrow$ $APL > MP_L$ [from (1) we get, $MP_L = \frac{W}{P}$]

Therefore a rational producers will always produce in stage II.

Laws of Returns to Scale

~~(Suppose that all the)~~

Returns to scale is a long run concept where both the inputs can be changed simultaneously. A change in scale means proportionate change in all inputs. Returns to scale refer to the relation between the change in the output level and the change in the scale of the production process.

Suppose that, all the inputs in the production process are changed in a certain proportion. If the output changes in the same proportion, we say that we have constant returns to scale. In this case, if all the inputs are doubled, output will be doubled; if the inputs are halved, output also will be halved. If when all the are changed in a certain proportion, and the change in output is in a smaller proportion, we have diminishing returns to scale. Here if all the inputs are doubled, output will increase but will not be doubled. Similarly, if when all the inputs are changed in a certain proportion and the change in output is in a larger proportion we have increasing returns to scale. Here, if all the inputs are doubled, the new level of output will be more than double of the old.

Let us consider the following pdn func.

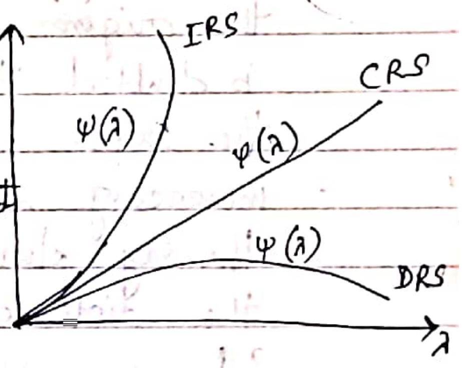
$$X = f(K, L)$$

Now this pdn func. is said to be homogenous if increase in all inputs by λ times $(\lambda > 0)$, leads to an

increase in the output by t^n times.
Here $f(\lambda K, \lambda L) \Rightarrow \lambda^n \cdot f(K, L)$

So this pdⁿ func. is homogenous of degree r.
Now if $r = 1$, ~~we~~ it would imply Constant Returns to Scale (CRS)
if $r > 1$, it would imply Increasing Returns to Scale (IRS)
and if $r < 1$, it means Decreasing Returns to Scale (DRS).

Mathematically there will be constant return to scale if the returns to scale function is a straight line. Under CRS, $\frac{dq}{d\lambda}$ is constant. Under IRS, $\frac{dq}{d\lambda}$ increases as λ increases. Under DRS, $\frac{dq}{d\lambda}$ decreases as λ increases. In the case of IRS, the returns to scale function is convex from below while under decreasing returns to scale (DRS), ~~($\frac{dq}{d\lambda}$ decreases as λ increases)~~ the returns to scale function is concave from below. This is shown in the fig.



Different types of returns to scale can also be represented with the help of the concept of scale elasticity of output. The scale elasticity of output is defined as

$\epsilon = \frac{dq/q}{d\lambda/\lambda}$. It is the percentage change in output due to 1% change in scale. If $\epsilon = 1$, the percentage change in output is equal to the percentage change in scale. In this case there will be CRS, Similarly, under IRS, $\epsilon > 1$ and under DRS, $\epsilon < 1$.

Graphical Representation of Different Types of Returns to Scale in an Isoquant Diagram –
In the case of CRS the distance between successive multiple

isoquants is constant along any ray through the origin as shown in the fig (a). In the fig OA = AB = BC. Thus when the length of the ray is doubled the output level is doubled. When the length of the ray is trebled, the output level is also trebled. The output level is proportional to the length of the ray.

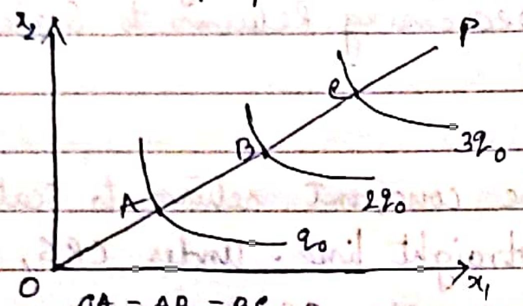


Fig (a)

In the case of IRS, the distance between consecutive multiple isoquants decreases on any ray through the origin as shown in fig (b). If the ray OA is doubled then the inputs are doubled and hence the output level is more than doubled because of increasing returns to scale. Hence to get doubled output the ray should be less than doubled. For this reason, the distance AB is less than the distance OA.

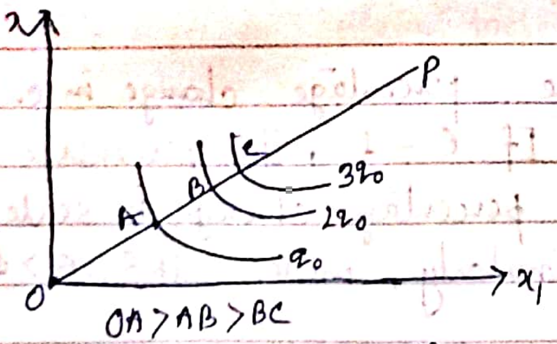
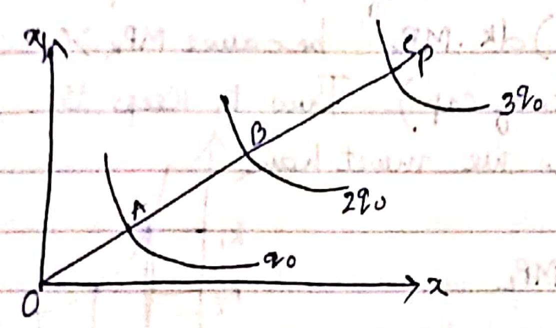


Fig (b)

In the case of DRS the distance between consecutive multiple isoquants increases along any ray through the origin as shown in the fig (c). Since there are DRS, the doubling of ray or doubling of inputs leads to less than doubled output. Hence to get doubled output the ray should be more than doubled. This means that $AB > OA$



$OA < AB < BC$

DRS

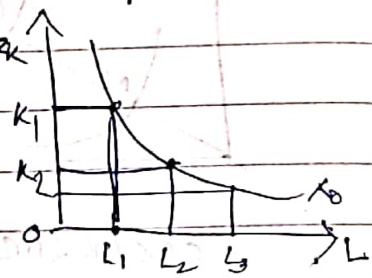
Fig (c)

Production with two variable inputs : Concept of Isoquants.

Let us consider the following pdn function :

$X = f(K, L)$ where K = capital, L = Labour and X = output
Now different combinations of K and L, which result in the same level of output, can be represented by an iso-quant or iso-product curve.
An iso-quant possesses the following properties :
1) The Iso-quant is negatively sloped - This implies that

if more of lab. (L) is employed in the production process, then less of cap. (K) has to be employed to keep the level of pdn the same as before. Here we assume that the marginal product of labour (MP_L) and marginal product of capital (MP_K) are positive. We also assume that one factor can be substituted for the other (though they are not perfect substitutes). Now, if more of L is employed, then it leads to an increase in pdn. by an amount $(dL \cdot MP_L)$, where dL = change in the amount of lab. So to keep the level of output the same as before, the firm should employ less of cap (K). Now, with the fall in the use of K, the output should fall by an amount $(-) dK \cdot MP_K$ because $MP_K > 0$. (dK = change in the amt. of cap.) Thus to keep the output level unchanged, we must have

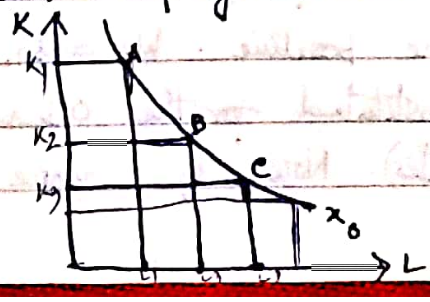


$$(-) dK \cdot MP_K = (+) dL \cdot MP_L$$

$$\text{or, } \frac{dK}{dL} = -\frac{MP_L}{MP_K} < 0$$

where $\frac{dK}{dL}$ = slope of the iso-quant. Hence the iso-quant must be negatively sloped.

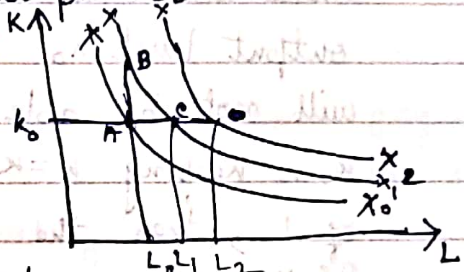
(ii) <u>The iso-quant is convex to the origin</u> – When the firm employs more lab. (L) and less cap. (K), then at each stage, it may become more difficult to substitute L for K because, they cannot be perfect substitutes, i.e all the work of cap. cannot be (~~substituted~~) (~~done~~) suitably done by the workers. Now the number of units of (cap) K_0, which can be replaced by one additional unit of lab. (L), is called the marginal rate of technical substitution of L for K, and it is denoted by $MRTS_{L,K}$ i.e. the slope of the isoquant. As L and K are assumed to be imperfect substitutes, so $MRTS_{L,K} = \left|\frac{dK}{dL}\right| = \frac{MP_L}{MP_K}$ declines gradually as the firm employs more and more of L.



In the fig. we observe that a movement along the isoquant from pt. A to B or from B to C, leads to a fall in the slope of the isoquant. Hence the isoquant becomes convex to the origin.

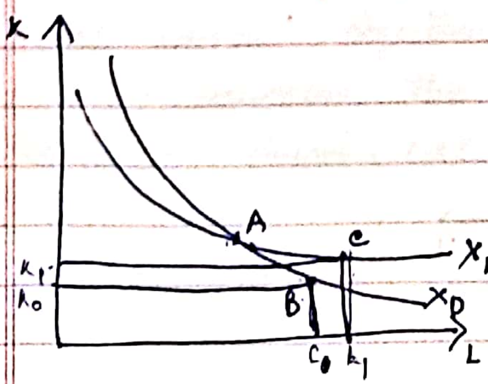
(iii) Any iso-quant situated above and to the right of another would indicate higher level of output.

In the fig., the combination pt. A on the isoquant XX_0 shows the input combination (K_0, L_0). Now keeping $L = L_0$ unchanged, if the firm employs more of K then output will rise because cap. has a positive marginal product. Thus a movement from pt. A on the isoquant XX_0 to the pt. B on the isoquant XX_1 would indicate a higher level of output. Following similar arguments, we can show that a movement from A to C would also increase the level of output. Thus the XX_1 would indicate a higher level of output as compared to the isoquant XX_0. So, in fig. $XX_0 < XX_1 < XX_2$.



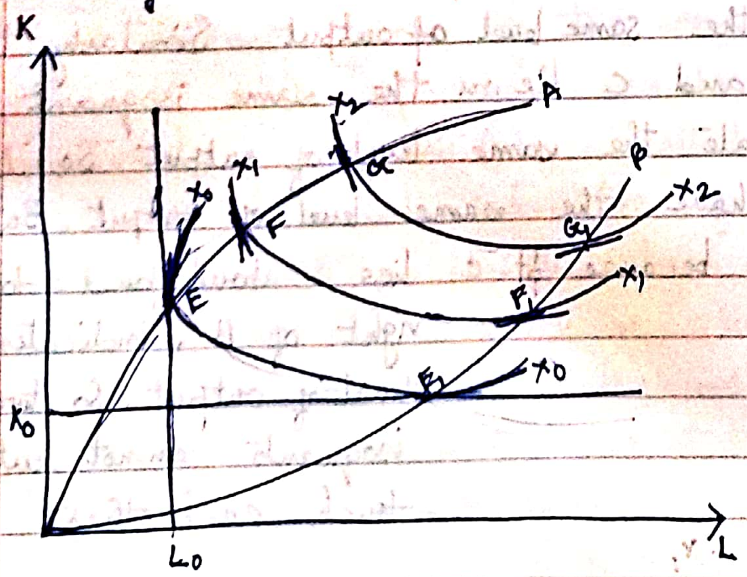
(iv) Two iso-quants cannot cut or touch one another -

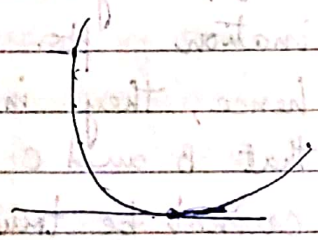
In the fig. two isoquants XX_0 and XX_1 have cut one another at pt. A. As both A and B points lie on the same isoquant XX_0, they show the same level of output. Similarly the combination pts. A and C lie on the same isoquant XX_1 and hence, they indicate the same level of output. So, it follows that B and C have the same level of output. But this cannot be true, because pt. C lies above and to the right of B, indicating higher level of output. So two isoquants can not cut or touch each other.



Iso-quants and the Ridge lines

In any pdn process, we need a given minimum level of K and L (say K_0 & L_0) to produce a given level output (say X_0). In fig, as we move along the iso-quant X_0X_0 from pt. E towards E_1, the firm employs more of L and less of K. However, if the minimum amount of K required for producing the output level X_0 is denoted by K_0, then the firm will not be able to reduce the amt. of K below K_0. Keeping $K = K_0$ constant, if the firm employs more of L, then the MP_L would be zero. Here, $MP_L = 0$ at pt. E_1. If the firm goes on employing more of L, then MP_L will fall, i.e. $MP_L < 0$ (following the law of diminishing returns to the variable factor during the S.R.). In this situation, the slope of the iso-quant, $\frac{dK}{dL} = (-)\frac{MP_L}{MP_K} > 0$, since $MP_L < 0$ and $MP_K > 0$. Thus, the isoquant X_0X_0 becomes positively sloped beyond the pt. E_1. This implies that the firm would have to employ more of both K and L to keep the level of output unchanged.





Similarly, a movement along the isoquant X_0X_0 from pt. E_1 towards the pt. E, leads to a fall in the employment of L and rise in the (pt) amt. of K to produce the output level X_0. Now, let us assume that $L = L_0$, is the minimum amount of L required to produce that level of output. Thus keeping $L = L_0$ unchanged, if the firm employs more of K, then $MP_K = 0$ at pt. E. If the firm still goes on employing more of K, then MP_K will fall, i.e. $MP_K < 0$. Thus, $\frac{dK}{dL} = (-) \frac{MP_L}{MP_K} > 0$. So beyond the pt. E, the isoquant X_0X_0 becomes positively sloped. Generally the firm avoids these positively sloped regions of the iso-quants, because it has to employ more of both K and L to produce same level of output. Now, in the fig., when we consider different iso-quants like X_0X_0, X_1X_1, X_2X_2 etc. then by joining all such pt.s like E, F and G where $MP_K = 0$, we get the ridge line OA. On the other hand by joining the pt.s like E_1, F_1 and G_1, where $MP_L = 0$, we get the ridge line OB. Thus, along any ridge lines, either MP_L or MP_K will be zero. If $MP_L = 0$, then $\frac{dK}{dL} = (-) \frac{MP_L}{MP_K} = 0$ along the ridge line OB. Again if $MP_K = 0$, then $(-) \frac{MP_L}{MP_K} = \infty$ along the ridge line OA. The firm generally chooses any input combinations within these ridge lines to produce any given level of output.

Producer's Equ^n.

The main objective of a producer is
i) to maximise the level of output, subject to a given cost constraint
or ii) to minimise cost, subject to a given output constraint.

Let us assume that w = given wage rate
r = given rental or the price of cap.
C = given amt. of money to be spent by the firm.

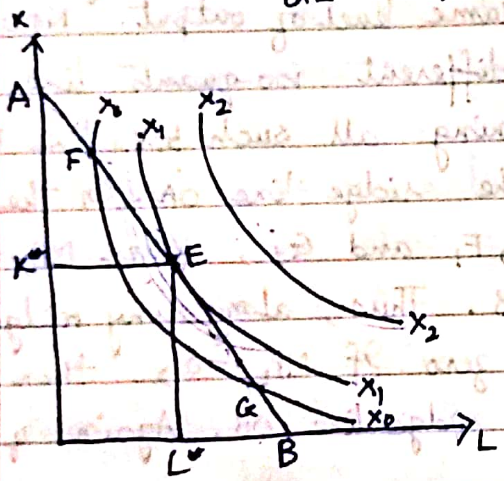
Maximise Level of output subject to a given cost constraints

$\therefore C = W.L + r.K$

where W.L denotes the cost of lab. and r.K denotes the cost of cap. This cost eqn. can also be expressed as:

$$\frac{C}{r} = \frac{W}{r} L + K \quad \text{or} \quad K = \frac{C}{r} - \frac{W}{r} . L$$

Now diff. combinations of K and L, which the firm can employ with the given amt. of money (C) and the prices of lab. (W) and cap. (r), can be represented by an isocost line. The slope of this isocost line will be $\frac{dK}{dL} = \frac{W}{r}$. In the fig. AB is the iso-cost line. The firm can employ any combination within the domain of the iso-cost line or any pt. on the iso-cost line, but not beyond the line. Thus given the cost constraint, the producer wants to achieve the maximum level of output.



Here x_0x_0, x_1x_1, x_2x_2 are different isoquants showing diff. level of output. If the firm chooses the input combination F, then any movement along the iso-cost line to the right of F, would mean a higher level of output with the same cost as before. Similarly the firm can move on to a higher levels of isoquants if it moves along the iso-cost line to the left of G. At pt. F, $\frac{-MP_L}{MP_K} > -\frac{W}{r}$ or, $\frac{MP_L}{W} > \frac{MP_K}{r}$. So his tendency would be employ more of L and less of K. Hence he would move along the iso-cost line from pt. F towards E.

Similarly, at pt. G, $\frac{MP_L}{MP_K} < \frac{w}{r}$ or, $\frac{MP_L}{w} < \frac{MP_K}{r}$. In this case firm would employ more of K and less of L. Such reallocation would continue until the producer reaches the highest possible iso-quant, given his cost constraint. This is known as the equ^m^ situation, where the producer would find the least-cost combination of factor inputs (say K^* and L^*). In the fig - equ^m^ pt. is denoted by pt. E, where

$$\frac{MP_L}{MP_K} = \frac{w}{r} \quad \text{or}, \quad \frac{MP_L}{w} = \frac{MP_K}{r}$$

Thus, the necessary or first order cond^n^ for such equ^m^ is the equality between the slope of the isoquant and the isocost line at equ^m^ pt. The sufficient or the 2nd order cond^n^ for this equ^m^ is that the $MRTS_{L,K}$ should be falling i.e. the isoquant is convex to the origin at the equ^m^ pt.

We can explain this mathematically with the help of Lagrangian multiplier (λ). Thus this constrained optimization problem can be represented as follows:

$$Z = f(K,L) + \lambda(C - w.L - rK)$$

Now differentiating this equ^n^ partially with respect to K, L and λ and setting them equal to zero, we get

$$\frac{\partial Z}{\partial L} = \frac{\partial f}{\partial L} - \lambda w = 0 \quad \text{.....(1)}$$

$$\frac{\partial Z}{\partial K} = \frac{\partial f}{\partial K} - \lambda r = 0 \quad \text{.....(2)}$$

$$\frac{\partial Z}{\partial \lambda} = C - w.L - r.K = 0 \quad \text{.....(3)}$$

From eq^n^ (1) & (2), we get

$$\frac{\partial f}{\partial L} = \lambda w \quad \text{and} \quad \frac{\partial f}{\partial K} = \lambda r$$

$$\Rightarrow \frac{MP_L}{\lambda} = w \quad \text{and} \quad \frac{MP_K}{\lambda} = r$$

$$\frac{\delta f}{\partial L} = \lambda w \Rightarrow MP_L = \lambda w$$

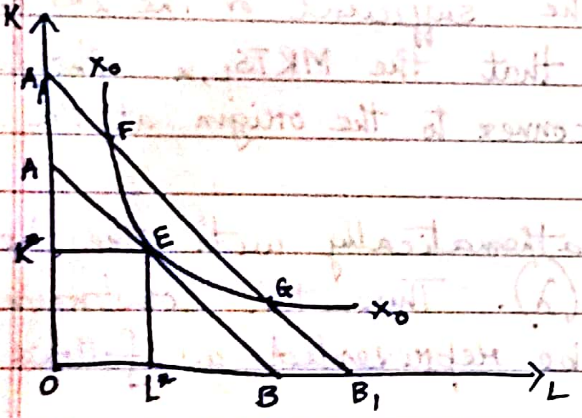
&

$$\frac{\delta f}{\delta K} = \lambda r \Rightarrow MP_K = \lambda r$$

$\Rightarrow \frac{MP_L}{MP_K} = \frac{W}{r}$ and this is our necessary condition.

Minimise cost subject to a given output constraint

The objective of the producer is to minimise cost, subject to a given level of output. In the fig. AB, A_1B_1 are different isocost lines having the same slope ($\frac{w}{r}$ ratio). However, the isocost line A_1B_1 represents higher total cost (c) in comparison with the iso-cost line AB. The vertical intercept (OA) shows the maximum amt. of K that the firm can employ with the given c and price of cap. Similarly the horizontal intercept shows the max. amt. of L that the firm can employ with the given amt. of C and the price of lab. Now in the fig. firm either moves from pt. F toward E, then it can reduce the total cost of pdn. for producing the same level of output (X_0). At pt. F, we see that the slope of the isoquant is greater than that of the isocost line i.e.



$$\frac{MP_L}{MP_K} > \frac{W}{r} \Rightarrow \frac{MP_L}{W} > \frac{MP_K}{r}$$

Hence the firm employs more of L and less of K. Through such reallocation of factor inputs, it can reduce its tatal cost of pdn. for producing the same level of output, X_0

Similarly, at pt. G, we see that

$$\frac{MP_L}{MP_K} < \frac{W}{r} \Rightarrow \frac{MP_L}{W} < \frac{MP_K}{r}$$

So in this case firm wants to employ more of K and less of L. This process continues until it reaches point E, where $\frac{MP_L}{MP_K} = \frac{W}{r}$ i.e. the pt. of tangency between the isoquant and isocost line. Thus pt. E is said to be the eqm pt. which shows the least-cost combination of factor inputs (K^*, L^*)

Thus necessary condn. → slope of isoquant = slope of isocost line
Sufficient condn. → $MRTS_{L,K}$ should be falling i.e. isoquant is convex to the origin

This problem can also be stated in the form of a Lagrangian function. Here the problem of the producer is to

$$\min.\ C = W.L + r.K \quad s.to \quad X_0 = f(K, L)$$

Lagrangian func. will be

$$Z = W.L + r.K - \lambda(X_0 - f(K, L))$$

Now differentiating this eqn partially w.r. to L, K and λ, and setting them equal to zero, we get

$$\frac{\delta Z}{\delta L} = W - \lambda \frac{\delta f}{\delta L} = 0 \quad \text{---(1)}$$

$$\frac{\delta Z}{\delta K} = r - \lambda \frac{\delta f}{\delta K} = 0 \quad \text{---(2)}$$

$$\frac{\delta Z}{\delta \lambda} = X_0 - f(K, L) = 0 \quad \text{...(3)}$$

From eqn (1) & (2) we get

$$W = \lambda \frac{\delta f}{\delta L} \Rightarrow \lambda \frac{\delta f}{\delta L} = W \Rightarrow \lambda . MPL = W$$

$$r = \lambda \frac{\delta f}{\delta K} = \lambda . \frac{\delta f}{\delta K} = r \Rightarrow \lambda MP_K = r$$

Now, $\frac{\lambda MP_L}{\lambda MP_K} = \frac{W}{r} \Rightarrow \frac{MP_L}{MP_K} = \frac{W}{r}$ which shows ~~that~~ the 1st order condn. for such constrained cost minimisation.

Homogenous Production Function

A function is said to be homogenous of degree r, if multiplication of each of its independent variables by any constant term, say λ, will change the value of the function by the proportion λ^r, i.e. if

$$f(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^r f(x_1, x_2, \dots x_n)$$

Let us consider the following pdn functions:

(a) $X = K^2 + L^2$

$$(\lambda K)^2 + (\lambda L)^2 = \lambda^2 (K^2 + L^2) = \lambda^2 . X$$

So this func. is homogenous of degree 2.

(b) $$X = \frac{\sqrt{K^2+L^2}}{K+L} = \frac{\sqrt{(\lambda K)^2 + (\lambda L)^2}}{\lambda K + \lambda L} = \frac{\sqrt{\lambda^2 (K^2+L^2)}}{\lambda (K+L)}$$

$$= \frac{\lambda \sqrt{K^2+L^2}}{\lambda (K+L)} = \lambda^0 X$$

In this case pdn func. is homogenous of degree 0.

(c) $X = \sqrt{K.L}$

$$\sqrt{\lambda K. \lambda L} = \sqrt{\lambda^2 (K.L)} = \lambda \sqrt{K.L} = \lambda . X$$

So here the pdn func. is homogenous of degree 1.

Linearly homogenous production function:

When the pdn func. is homogenous of degree 1, we call it a linearly homogenous pdn func.

Properties:

① In the case of any linearly homogenous pdn func. say $X = f(K, L)$ the APL $= \frac{X}{L}$ and APK $= \frac{X}{K}$

can be expressed as functions of cap.-lab. ratio $\left(\frac{K}{L}\right)$ alone.

Let $\lambda = \frac{1}{L}$, So we get $\lambda X = f(\lambda K, \lambda L)$

$$\Rightarrow \frac{X}{L} = f\left(\frac{K}{L}, 1\right)$$

$$\Rightarrow AP_L = f\left(\frac{K}{L}\right)$$

~~Let $\lambda = \frac{1}{K}$, So we get $\lambda X = f(\lambda K, \lambda L)$~~

~~$\frac{X}{K} = f\left(1, \frac{L}{K}\right)$~~

OMIT

~~$\Rightarrow AP_K = f\left(\frac{L}{K}\right)$~~

We know $AP_K = \frac{X}{K} = \frac{X}{L} \cdot \frac{L}{K}$

$$= f\left(\frac{K}{L}\right) \Big/ \frac{K}{L}$$

Hence both AP_L and AP_K depend only on the K/L ratio.

(ii) In the case linearly homogenous pdn func., both ~~(the marginal)~~ MP_L and MP_K would also be the func. of $\left(\frac{K}{L}\right)$ ratio

Since $\frac{X}{L} = f\left(\frac{K}{L}\right)$ $\therefore X = L \cdot f\left(\frac{K}{L}\right)$

$$\therefore MP_L = \frac{\partial X}{\partial L} = \frac{\partial}{\partial L}\left[L \cdot f\left(\frac{K}{L}\right)\right]$$

$$= f\left(\frac{K}{L}\right) + L \cdot f'\left(\frac{K}{L}\right) \cdot \left(\frac{-K}{L^2}\right) \quad \left[\because \frac{\partial K}{\partial L} = 0\right]$$

$$= f\left(\frac{K}{L}\right) - f'\left(\frac{K}{L}\right) \cdot \left(\frac{K\cancel{L}}{L^{\cancel{2}}}\right)$$

$$= f\left(\frac{K}{L}\right) - f'\left(\frac{K}{L}\right) \cdot \left(\frac{K}{L}\right)$$

$$= g\left(\frac{K}{L}\right).$$

Thus MP_L is a function of K/L ratio.

Similarly $MP_K = \frac{\partial X}{\partial K} = \frac{\partial}{\partial K}\left(L \cdot f\left(\frac{K}{L}\right)\right) = \frac{\partial L}{\partial K} \cdot f\left(\frac{K}{L}\right) + L \cdot f'\left(\frac{K}{L}\right)\left(\frac{L}{L^2}\right)$

$$= f'\left(\frac{K}{L}\right)\frac{L^2}{L^2} = f'\left(\frac{K}{L}\right) \quad \left[\because \frac{\partial L}{\partial K} = 0\right]$$

Thus MP_K is also a function of the $\frac{K}{L}$ ratio.

(iii) If each factor is multiplied by its MP, then their sum would be equal to the TP, ie.

$$K \cdot \frac{\delta x}{\delta K} + L \cdot \frac{\delta x}{\delta L} = x$$

Proof :

Here $\frac{\delta x}{\delta K} = f'\left(\frac{K}{L}\right)$ and $\frac{\delta x}{\delta L} = f\left(\frac{K}{L}\right) - f'\left(\frac{K}{L}\right) \cdot \frac{K}{L}$

$\therefore K \cdot f'\left(\frac{K}{L}\right) + L \cdot f\left(\frac{K}{L}\right) - \cancel{L} \cdot f'\left(\frac{K}{L}\right) \cdot \frac{K}{\cancel{L}}$

$\Rightarrow = K \cdot f'\left(\frac{K}{L}\right) + L \cdot f\left(\frac{K}{L}\right) - K \cdot f'\left(\frac{K}{L}\right)$

$= L \cdot f\left(\frac{K}{L}\right) = X$ [Since, total output is expressed as $X = L \cdot f\left(\frac{K}{L}\right)$]

Cobb-Douglus Production Function

C.W. Cobb and P. H. Douglus proposed a pdn. func. of the following form :

$$X = A \cdot K^{\alpha} \cdot L^{\beta}$$

where A = technological parameter

K = Cap., L = Lab.,

α = elasticity of output with respect to cap.

β = elasticity of output with respect to lab.

Special features:

i) This pdn. func. is homogenous of degree $(\alpha+\beta)$. In the special case, when $\alpha+\beta = 1$, it signifies a linearly homogenous pdn func.

(ii) Its iso-quants are negatively sloped throughout and they are also strictly convex to the origin for positive values of K and L.

(iii) Its exponents α and β represent:
(a) elasticity of output with respect to K and L respectively
(b) share of each input in total output.

Properties

1/ The Cobb-Douglus pdn func. is homogenous of degree $(\alpha+\beta)$, because if K and L are increased by λ proportion, then the output would increase by $\lambda^{\alpha+\beta}$ proportion.

Here $A(\lambda K)^{\alpha}(\lambda L)^{\beta} = A\lambda^{\alpha+\beta}K^{\alpha}L^{\beta} \Rightarrow \lambda^{\alpha+\beta}\cdot AK^{\alpha}L^{\beta} = \lambda^{\alpha+\beta}\cdot X$

If $\alpha+\beta = 1$, then this function will be linearly homogenous. In this case $X = A\cdot K^{\alpha}L^{1-\alpha}$

2/ In the case of Cobb-Douglus pdn func., the isoquants are negatively sloped.
Here $X = AK^{\alpha}L^{\beta}$,
Along any isoquant, total output remains unchanged. $\therefore dX = 0$

$$dX = \frac{\delta X}{\delta K}\cdot dK + \frac{\delta X}{\delta L}\cdot dL = 0$$

$$\Rightarrow (A\cdot\alpha K^{\alpha-1}\cdot L^{\beta})\,dK + (A\cdot\beta\cdot K^{\alpha}L^{\beta-1})\,dL = 0 \qquad \left[MP_L = \frac{\delta X}{\delta L},\; MP_K = \frac{\delta X}{\delta K}\right]$$

$$\Rightarrow (A\alpha K^{\alpha-1}L^{\beta})\,dK = -(A\cdot\beta\cdot K^{\alpha}L^{\beta-1})\,dL$$

$$\Rightarrow \frac{dK}{dL} = \boxed{-\frac{\cancel{A\alpha}K^{\alpha-1}L^{\beta}}{A\beta K^{\alpha}L^{\beta-1}} = -\left(\frac{\alpha}{\beta}\right)\left(\frac{L}{}\right)} = -\frac{\cancel{A}\beta K^{\alpha}L^{\beta-1}}{\cancel{A}\alpha K^{\alpha-1}L^{\beta}} = (-)\left(\frac{\beta}{\alpha}\cdot\frac{K}{L}\right) < 0$$

So the isoquants (i.e. $\frac{dK}{dL}$) will be negatively sloped.

These isoquants would be convex to the origin, i.e. $\frac{d^2K}{dL^2} > 0$

Here $\frac{d^2K}{dL^2} = \frac{d}{dL}\left(-\frac{\beta}{\alpha}\cdot\frac{K}{L}\right)$

$$= -\frac{\beta}{\alpha}\cdot\frac{d(K/L)}{dL} = -\frac{\beta}{\alpha}\cdot\left[\frac{1}{L}\cdot\frac{dK}{dL} - K\cdot\frac{1}{L^2}\right]$$

$$= (-)\frac{\beta}{\alpha}\cdot\frac{1}{L}\cdot\frac{dK}{dL} + \frac{\beta}{\alpha}\cdot\frac{K}{L^2} > 0 \quad \because \frac{dK}{dL} < 0$$

3/ The elasticity of output with respect to lab. (e_L) shows the response in output with respect to one percent change in lab. inputs. This is expressed as follows:

$$e_L = \frac{\frac{dX}{X}}{\frac{dL}{L}} = \frac{dX}{dL}\cdot\frac{L}{X}$$

$$= A\beta K^{\alpha} L^{\beta-1}\cdot\frac{L}{A\cdot K^{\alpha}L^{\beta}}$$

$$= \beta\cdot K^{\alpha-\alpha}\cdot L^{\beta-1-\beta+1}$$

$$= \beta$$

So the exponent β represents the elasticity of output with respect to lab. inputs.

Further, this elasticity implies the relative share of total output accruing to labour.

Here $e_L = \frac{dX}{dL}\cdot\frac{L}{X} = L\cdot\frac{\frac{dX}{dL}}{X} = \beta$ $\left[\frac{dX}{dL} = MP_L \text{ (lab. } \cancel{\text{payment}} \text{ to be paid according to its } MP_L\text{)}\right.$

$\left.\therefore L\cdot MP_L = \text{Income of lab.}\right]$

Thus, the share of labour's real income in total output (X) will be β.

Similarly, the exponent α in the Cobb-Douglas pdn func. shows the output elasticity with respect to cap. (e_K).

Thus, $e_K = \frac{\frac{dX}{X}}{\frac{dK}{K}} = \frac{dX}{dK} \cdot \frac{K}{X}$

$= \cancel{A}\alpha K^{\alpha-1} L^{\beta} \cdot \frac{K}{\cancel{A}K^{\alpha} L^{\beta}} = \alpha . K^{\alpha-1+\alpha+1} . L^{\beta-\beta}$

$= \alpha$

Again $e_K = \frac{dX}{dK} \cdot \frac{K}{X} = K . \frac{dX/dK}{X} = \frac{K.MP_K}{X} = \alpha$

∴ the relative share of total output accruing to cap. is α.

4/ The pdⁿ func. is such that if the quantity of one (input) of the factors employed is zero, then total output is also zero. This means that both the factors are indispensible for producing output. We can get positive output only if both the inputs are used in positive quantities.

5/ It can be seen that MP_L and MP_K depend on the ratio of two factors.

Since $X = AK^{\alpha} . L^{1-\alpha}$ $[\alpha+\beta=1]$

$\therefore \frac{\partial X}{\partial L} = (1-\alpha) . A K^{\alpha} . L^{-\alpha}$

$= (1-\alpha) . A . \left(\frac{K}{L}\right)^{\alpha}$

$\frac{\partial X}{\partial K} = \alpha A . K^{\alpha-1} L^{1-\alpha}$

$= \alpha . A . K^{\alpha-1} . L^{-(\alpha-1)}$

$= \alpha . A . \left(\frac{K}{L}\right)^{\alpha-1}$

6/ The elasticity of substitution for the Cobb-Douglas pdⁿ func. is equal to unity.

$$\sigma = \frac{d \log\left(\frac{K}{L}\right)}{d \log (MRTS_{(L,K)})}$$

$$MRTS_{(L,K)} = \frac{\partial X/\partial L}{\partial X/\partial K}$$

$$= \frac{\cancel{A\alpha K}\ A(1-\alpha)K^{\alpha}L^{1-\alpha-1}}{A.\alpha.K^{\alpha-1}L^{1-\alpha}} \quad [X = A.K^{\alpha}L^{1-\alpha}]$$

$$\cancel{= 1 - \alpha \quad \alpha . K} \quad = \frac{(1-\alpha)\cancel{A}.\left(\frac{K}{L}\right)^{\alpha}}{\alpha.\cancel{A}\left(\frac{K}{L}\right)^{\alpha-1}}$$

$$= \frac{1-\alpha}{\alpha}\left(\frac{K}{L}\right)^{\cancel{\alpha}-\cancel{\alpha}+1}$$

$$= \frac{1-\alpha}{\alpha}.\frac{K}{L}$$

$$\log(MRTS_{L,K}) = \log\left(\frac{1-\alpha}{\alpha}\right) + \log\left(\frac{K}{L}\right)$$

$$= \log$$

$$\Rightarrow \log(K/L) = \cancel{\log\left(\frac{1-\alpha}{\alpha}\right)} \log(MRTS_{L,K}) - \log\left(\frac{1-\alpha}{\alpha}\right)$$

Now. $$\frac{d\log(K/L)}{d\log MRTS_{L,K}} = 1$$

or, $\sigma = 1$

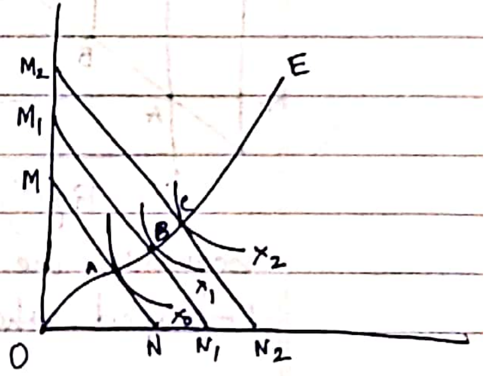
Expansion Path

The expansion path of a firm shows the expansion of its optimum level of output with the employment of least-cost factor combinations. The necessary condn for producer's eqm is $\frac{dK}{dL} = -\frac{MP_L}{MP_K} = -\frac{w}{r}$

Here, $\left|\frac{dK}{dL}\right| = MRTS_{L,K} \Rightarrow \frac{MP_L}{MP_K} = \frac{w}{r}$

which implies that the slope of the isoquant has to be equal to that of the iso-cost line at the eqm pt. ~~The~~

The sufficient condn requires the strict convexity of the iso-quant (i.e. diminishing $MRTS_{L,K}$) at the eqm pt. Now joining all such ~~cost~~ eqm pt.s (such as A, B, C in fig), we get the expansion path (OE). In the fig. X_0, X_1 and X_2 represent diff. isoquants showing diff. output levels ($X_2 > X_1 > X_0$) and MN, M_1N_1 and M_2N_2 are three iso-cost lines. These isocost lines have the same slope implying same $\frac{w}{r}$ ratio. Each pt. on the expansion path gives us a particular level of output, a least cost factor combination to produce that level of output and the total cost of employing these factor inputs.



Pdn Function: Two Special Cases

Two extreme cases of pdn func. show the possible range of input substitution in the pdn process:

1) In fig (1) inputs to pdn are perfect substitute for one another. Here the MRTS is constant at all pt.s on an isoquant

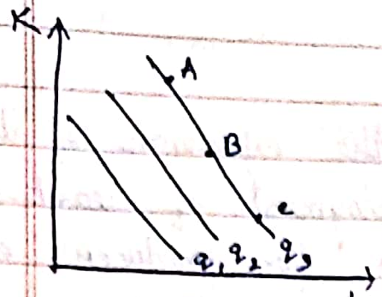


Fig (1)

As a result, the same output (say q_3) can be produced with mostly cap. (at A), with mostly lab. (at C) or with a balanced combination of both (at B).

Here the rate at which cap. and lab. can be substituted for each other is the same no matter what level of inputs is being used.

Fig (2) shows the opposite extreme, the fixed-proportions pdn func., sometimes called a Leontief pdn func. In this case, it is impossible to make any substitution among inputs. Each level of output requires a specific combination of labour and cap: Additional output cannot be obtained unless more cap. and lab. are added in specific proportions. As a result, the

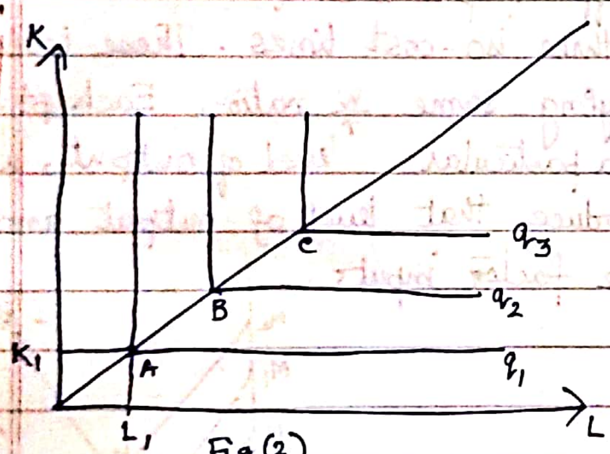


Fig (2)

isoquants are L-shaped, just as indifference curve are L-shaped when (~~the~~) two goods are perfect compliments.

<u>The elasticity of substitution</u>

As the price ratio P_L/P_K of inputs changes, the slope of the isocost line changes, and we get a new pt. of tangency and a new level of input usage for lab. and cap. Thus the L/K ratio falls. There is a measure of responsiveness of L/K to a change in P_L/P_K. This is called the elasticity of substitution (σ).

$$\sigma = \left| \frac{\text{Percent change in } L/K}{\text{percent change in } P_L/P_K} \right|$$